Kishal Tandel

Chapter 1

August 19, 2025

Classical Mechanics

Third Edition

Herbert Goldstein, Charles Poole, and John Safko

Index

Problem 1 .... **3**

Problem 2 .... **3**

Problem 3 .... **5**

Problem 4 .... **7**

Problem 5 .... **9**

Problem 6 .... **11**

Problem 7 .... **12**

Problem 8 .... **13**

Problem 9 .... **14**

Problem 10 .... **15**

Problem 11 .... **16**

Problem 12 .... **16**

Problem 13 .... **17**

Problem 14 .... **20**

Problem 15 .... **21**

Problem 16 .... **22**

Problem 17 .... **23**

Problem 18 .... **24**

Problem 19 .... **26**

Problem 20 .... **27**

Problem 21 .... **28**

Problem 22 .... **29**

Problem 23 .... **31**

Problem 24 .... **32**

Problem 1

Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

Solution 1

If mass is constant, the equation of motion is:

$$\mathbf{F} = m\dot{\mathbf{v}} \tag{1.1}$$

Taking dot product of Eq. (1.1) with $\mathbf{v}$, we get:

$$\begin{aligned}\mathbf{F} \cdot \mathbf{v} &= m\dot{\mathbf{v}} \cdot \mathbf{v} \\ &= \frac{m}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) \\ &= \frac{m}{2}\frac{d}{dt}\|\mathbf{v}\|^2 \\ &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \boxed{\frac{dT}{dt}}\end{aligned}$$

If mass varies with time, the equation of motion is:

$$\mathbf{F} = \dot{\mathbf{p}} \tag{1.2}$$

Taking dot product of Eq. (1.2) with $\mathbf{p}$, we get:

$$\begin{aligned}\mathbf{F} \cdot \mathbf{p} &= \dot{\mathbf{p}} \cdot \mathbf{p} \\ &= \frac{d}{dt}(m\mathbf{v}) \cdot m\mathbf{v} \\ &= m\frac{dm}{dt}\|\mathbf{v}\|^2 + m^2\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} \\ &= \frac{d}{dt}\left(\frac{1}{2}m^2v^2\right) = \boxed{\frac{d(mT)}{dt}}\end{aligned}$$

Problem 2

Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i \neq j} m_i m_j r_{ij}^2$$

Solution 2

The position vector of center of mass is given by:

$$\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i \tag{2.1}$$

Taking dot product of Eq. (2.1) with itself, we get:

$$R^2 = \frac{1}{M^2}\left[\sum_i m_i^2 r_i^2 + \sum_{i\neq j} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j\right] \tag{2.2}$$

Note that $\mathbf{r_i}\cdot\mathbf{r_j} = \frac{1}{2}(r_i^2 + r_j^2 - r_{ij}^2)$. Hence,

$$\begin{aligned}
\sum_{i\neq j} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j &= \frac{1}{2}\left[\sum_{i\neq j} m_i m_j r_i^2 + \sum_{j\neq i} m_i m_j r_j^2 - \sum_{i\neq j} m_i m_j r_{ij}^2\right] \\
&= \frac{1}{2}\left[\sum_j m_j \sum_{i\neq j} m_i r_i^2 + \sum_i m_i \sum_{j\neq i} m_j r_j^2 - \sum_{i\neq j} m_i m_j r_{ij}^2\right] \\
&= \frac{1}{2}\left[2\sum_j m_j \sum_{i\neq j} m_i r_i^2 - \sum_{i\neq j} m_i m_j r_{ij}^2\right] \\
&= \sum_j m_j \sum_{i\neq j} m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2
\end{aligned} \tag{2.3}$$

Also, note that:

$$\sum_i m_i^2 r_i^2 = \sum_{i=j} m_j m_i r_i^2 = \sum_j m_j \sum_{i=j} m_i r_i^2 \tag{2.4}$$

Using Eqs. (2.3) and (2.4) in Eq. (2.2), we get:

$$\begin{aligned}
R^2 &= \frac{1}{M^2}\left[\sum_j m_j \sum_{i=j} m_i r_i^2 + \sum_j m_j \sum_{i\neq j} m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2\right] \\
&= \frac{1}{M^2}\left[\sum_j m_j \sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2\right] \\
&= \frac{1}{M^2}\left[M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2\right]
\end{aligned} \tag{2.5}$$

Hence using Eq. (2.5), we have:

$$\boxed{M^2 R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2}$$

Problem 3

Suppose a system of two particles is known to obey the equations of motion, Eqs. (1.22) and (1.26). From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to Eqs. (1.22) and (1.26).

Remark: By Eqs. (1.22) and (1.26) of the book Classical Mechanics, the author(s) is/are referring to

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F} \quad \text{and} \tag{3.1}$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} \quad \text{respectively.} \tag{3.2}$$

Solution 3

Consider a system of two particles with mass m_i and position vector $\mathbf{r}_i$ where $i \in \{1, 2\}$. The equation of motion for first particle is:

$$\mathbf{F}_{21} + \mathbf{F}_1^{(e)} = m_1\ddot{\mathbf{r}}_1 \tag{3.3}$$

Similarly, the equation of motion for second particle is:

$$\mathbf{F}_{12} + \mathbf{F}_2^{(e)} = m_2\ddot{\mathbf{r}}_2 \tag{3.4}$$

Adding Eqs. (3.3) and (3.4), we get:

$$\begin{aligned} \mathbf{F}_{21} + \mathbf{F}_{12} + \mathbf{F}_1^{(e)} + \mathbf{F}_2^{(e)} &= m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 \\ &= M\ddot{\mathbf{R}} \end{aligned} \tag{3.5}$$

where $M = (m_1 + m_2)$ and $\mathbf{R} = \dfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$. Using Eq. (3.1) in Eq. (3.5), we have:

$$\begin{aligned} \mathbf{F}_{21} + \mathbf{F}_{12} &= 0 \\ \Rightarrow \quad \boxed{\mathbf{F}_{21} = -\mathbf{F}_{12}} \end{aligned} \tag{3.6}$$

Hence the internal forces between particles satisfy the weak law of action and reaction.

Now, consider the cross product of $\mathbf{r}_1$ with Eq. (3.3) and $\mathbf{r}_2$ with Eq. (3.4).

$$\mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_1 \times \mathbf{F}_1^{(e)} = \mathbf{r}_1 \times m_1\ddot{\mathbf{r}}_1 \tag{3.7}$$

$$\mathbf{r}_2 \times \mathbf{F}_{12} + \mathbf{r}_2 \times \mathbf{F}_2^{(e)} = \mathbf{r}_2 \times m_2\ddot{\mathbf{r}}_2 \tag{3.8}$$

Adding Eqs. (3.7) and (3.8) and noting that $\forall\ i \in \{1, 2\},\ \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \mathbf{N}_i^{(e)}$ and $\sum_i \mathbf{N}_i^{(e)} = \mathbf{N}^{(e)}$ and then using Eq. (3.2), we get:

$$\mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12} + \dot{\boldsymbol{L}} = \mathbf{r}_1 \times m_1\ddot{\mathbf{r}}_1 + \mathbf{r}_2 \times m_2\ddot{\mathbf{r}}_2 \tag{3.9}$$

But $\dot{\boldsymbol{L}} = \sum\limits_i \frac{d}{dt}(\mathbf{r}_i \times m_i\dot{\mathbf{r}}_i) = \sum\limits_i \dot{\mathbf{r}}_i \times m_i\dot{\mathbf{r}}_i + \sum\limits_i \mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i = \sum\limits_i \mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i,\ \forall\ i \in \{1,2\}$. Using this fact in Eq. (3.9), we have:

$$\begin{aligned}
& & \mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12} = 0 & & \\
\Rightarrow \quad & (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{21} = 0 \quad \text{and} \quad & (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_{12} = 0 & \quad & \text{[Using Eq. (3.6)]} \\
\Rightarrow \quad & \mathbf{r}_{12} \times \mathbf{F}_{21} = 0 \quad \text{and} \quad & \mathbf{r}_{21} \times \mathbf{F}_{12} = 0 & \quad & \text{[Denoting } (\mathbf{r}_i - \mathbf{r}_j) \text{ as } \mathbf{r}_{ij}\text{]} \\
\Rightarrow \quad & \boxed{\mathbf{r}_{21} \times \mathbf{F}_{21} = 0} \quad \text{and} \quad & \boxed{\mathbf{r}_{12} \times \mathbf{F}_{12} = 0} & \quad & \text{[Using } \mathbf{r}_{ij} = -\mathbf{r}_{ji}\text{]}
\end{aligned}$$

Since $\mathbf{r}_{ij} \neq 0$ and $\mathbf{F}_{ij} \neq 0$, $\forall\, i,j \in \{1,2\}$ and $i \neq j$, we must have $\mathbf{F}_{ij} \parallel \mathbf{r}_{ij}$, $\forall\, i,j \in \{1,2\}$ and $i \neq j$. Hence the internal forces between particles satisfy the strong law of action and reaction too.

Now, consider a system with arbitrary number of particles. The equation of motion for any particle is:

$$\mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = m_i\ddot{\mathbf{r}}_i \qquad \forall\, i,j \text{ and } i \neq j \tag{3.10}$$

Eq. (3.10) represents as many equations of motion as there are particles in the system, one each for each particle. Summing all these equations, we get:

$$\sum_{i\neq j} \mathbf{F}_{ji} + \sum_i \mathbf{F}_i^{(e)} = \sum_i m_i\ddot{\mathbf{r}}_i \tag{3.11}$$

$$\Rightarrow \quad \sum_{i\neq j} \mathbf{F}_{ji} + \sum_i \mathbf{F}_i^{(e)} = M\ddot{\mathbf{R}} \tag{3.12}$$

where $M = \sum\limits_i m_i$ and $\mathbf{R} = \dfrac{\sum\limits_i m_i\mathbf{r}_i}{\sum\limits_i m_i}$. Note that (3.12) can be written as:

$$\frac{1}{2}\sum_{i\neq j}(\mathbf{F}_{ji} + \mathbf{F}_{ij}) + \sum_i \mathbf{F}_i^{(e)} = M\ddot{\mathbf{R}} \tag{3.13}$$

If the internal forces between the particles follow the weak law of action and reaction, i.e., $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$, $\forall\, i,j$ and $i \neq j$, we have using Eq. (3.13),

$$\boxed{\sum_i \mathbf{F}_i^{(e)} = M\ddot{\mathbf{R}}}$$

thus proving the converse of the generalized arguments leading to Eq. (3.1).

Now, consider the cross product of $\mathbf{r}_i$ with Eq. (3.11):

$$\sum_{i\neq j} \mathbf{r}_i \times \mathbf{F}_{ji} + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \sum_i \mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i \tag{3.14}$$

But $\sum\limits_i \mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i = \sum\limits_i \dot{\mathbf{r}}_i \times m_i\dot{\mathbf{r}}_i + \sum\limits_i \mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i = \sum\limits_i \frac{d}{dt}(\mathbf{r}_i \times m_i\dot{\mathbf{r}}_i) = \dot{\boldsymbol{L}}$ and $\sum\limits_i \mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i = \sum\limits_i \mathbf{N}_i^{(e)} = \mathbf{N}^{(e)}$. Using these in Eq. (3.14), we have:

$$\sum_{i\neq j}(\mathbf{r}_i \times \mathbf{F}_{ji}) + \mathbf{N}^{(e)} = \dot{\boldsymbol{L}} \tag{3.15}$$

Note that Eq. (3.15) is written for i^{th} particle. Similarly we can write the equation for j^{th} particle by changing the indices as follows:

$$\sum_{j\neq i}(\mathbf{r}_j \times \mathbf{F}_{ij}) + \mathbf{N}^{(e)} = \dot{\boldsymbol{L}} \tag{3.16}$$

Adding Eqs. (3.15) and (3.16), we get:

$$\sum_{i\neq j}(\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij}) + 2\mathbf{N}^{(e)} = 2\dot{\boldsymbol{L}} \tag{3.17}$$

If the internal forces between the particles follow the strong law of action and reaction, i.e., $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and $\mathbf{F}_{ij} \parallel \mathbf{r}_{ij}$, $\forall\, i, j$ and $i \neq j$, Eq. (3.17) can be written in the following ways:

$$\begin{aligned}
&\sum_{i\neq j}[(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}] + 2\mathbf{N}^{(e)} = 2\dot{\boldsymbol{L}} \quad \text{and} \quad \sum_{j\neq i}[(\mathbf{r}_j - \mathbf{r}_i) \times \mathbf{F}_{ij}] + 2\mathbf{N}^{(e)} = 2\dot{\boldsymbol{L}} \\
\Rightarrow \quad &\sum_{i\neq j}(\mathbf{r}_{ij} \times \mathbf{F}_{ji}) + 2\mathbf{N}^{(e)} = 2\dot{\boldsymbol{L}} \quad \text{and} \quad \sum_{j\neq i}(\mathbf{r}_{ji} \times \mathbf{F}_{ij}) + 2\mathbf{N}^{(e)} = 2\dot{\boldsymbol{L}} \qquad [\text{Denoting } (\mathbf{r}_i - \mathbf{r}_j) \text{ as } \mathbf{r}_{ij}] \\
\Rightarrow \quad &-\sum_{i\neq j}(\mathbf{r}_{ji} \times \mathbf{F}_{ji}) + 2\mathbf{N}^{(e)} = 2\dot{\boldsymbol{L}} \quad \text{and} \quad -\sum_{j\neq i}(\mathbf{r}_{ij} \times \mathbf{F}_{ij}) + 2\mathbf{N}^{(e)} = 2\dot{\boldsymbol{L}} \qquad [\text{Using } \mathbf{r}_{ij} = -\mathbf{r}_{ji}] \\
\Rightarrow \quad &\boxed{\mathbf{N}^{(e)} = \dot{\boldsymbol{L}}} \qquad [\because \mathbf{F}_{ij} \parallel \mathbf{r}_{ij}]
\end{aligned}$$

Hence we have proved the converse of the generalized arguments leading to Eq. (3.2).

Problem 4

The equations of constraint for the rolling disk, Eqs. (1.39), are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_n)dx_i = 0.$$

A constraint condition of this type is holonomic only if an integrating function $f(x_1, \ldots, x_n)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of Eqs. (1.39).

Remark: By Eqs. (1.39) of the book Classical Mechanics, the author(s) is/are reffering to

$$dx - a\sin\theta d\phi = 0 \qquad \text{and} \tag{4.1}$$

$$dy + a\cos\theta d\phi = 0 \tag{4.2}$$

Solution 4

A rigid disk in three dimensions has six degrees of freedom. Constraining it to move on a plane and requiring it's plane to be always vertical are both holonomic constraints as they can be expressed in the form $f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, t) = 0$ as follows:

$$z - z_o = 0, \qquad \text{where } z_o \text{ is a constant}$$
$$n_z = 0, \qquad \text{where } n_z \text{ is the component of disk's unit normal vector parallel to } z\text{-axis}$$

These constraints thus reduce the degrees of freedom of the system to four. The system can thus be described by the coordinates (x, y, θ, ϕ), where (x, y) denotes the disk's center; θ represents its orientation in the plane; and ϕ indicates the rotation about its axis. The constraint conditions given by Eqs. (4.1) and (4.2), in general, can be written as:

$$g_x dx + g_y dy + g_\theta d\theta + g_\phi d\phi = 0$$

In Eq. (4.1), $g_x = 1, g_y = 0, g_\theta = 0, g_\phi = -a\sin\theta$. We want to find $f(x, y, \theta, \phi)$ such that it satisfies $\binom{4}{2} = 6$ conditions as follows:

$$\begin{aligned}
\frac{\partial(fg_x)}{\partial y} &= \frac{\partial(fg_y)}{\partial x} \quad &\Rightarrow \quad & \frac{\partial f}{\partial y} = 0 \\
\frac{\partial(fg_x)}{\partial \theta} &= \frac{\partial(fg_\theta)}{\partial x} \quad &\Rightarrow \quad & \frac{\partial f}{\partial \theta} = 0 \qquad (4.3)\\
\frac{\partial(fg_x)}{\partial \phi} &= \frac{\partial(fg_\phi)}{\partial x} \quad &\Rightarrow \quad & \frac{\partial f}{\partial \phi} = -a\sin\theta\frac{\partial f}{\partial x} \\
\frac{\partial(fg_y)}{\partial \theta} &= \frac{\partial(fg_\theta)}{\partial y} \quad &\Rightarrow \quad & 0 = 0 \\
\frac{\partial(fg_y)}{\partial \phi} &= \frac{\partial(fg_\phi)}{\partial y} \quad &\Rightarrow \quad & 0 = -a\sin\theta\frac{\partial f}{\partial y} \\
\frac{\partial(fg_\theta)}{\partial \phi} &= \frac{\partial(fg_\phi)}{\partial \theta} \quad &\Rightarrow \quad & 0 = -a\sin\theta\frac{\partial f}{\partial \theta} - af\cos\theta \qquad (4.4)
\end{aligned}$$

Using Eq. (4.3) in Eq. (4.4), we conclude that there is no non-trivial integrating factor f unless $\theta = \pi/2$ throughout. Hence Eq. (4.1) cannot be written as an exact differential unless $\theta = \pi/2$ throughout.

If $\theta = \pi/2$ throughout, the disk is rolling parallel to x-axis; Eqs. (4.1) and (4.2) can then be integrated and the resulting relation between the coordinates (due to the constraint that the disk rolls without slipping) will be:

$$x - x_o = a(\phi - \phi_o), \qquad \text{where } x_o \text{ and } \phi_o \text{ are constants.}$$
$$y - y_o = 0, \qquad \text{where } y_o \text{ is a constant.}$$

which are holonomic equations of constraint. In this case, the degrees of freedom of the system further reduce by three (as we now have three more holonomic constraint equations, one each in θ, x and y). Hence we need just one generalized coordinate (a simple choice will be ϕ) to describe the system when θ is $\pi/2$ throughout.

But there doesn't exist any general integrating factor f such that Eq. (4.1) reduces to exact differential and hence Eq. (4.1) is nonholonomic differential constraint condition.

In Eq. (4.2), $g_x = 0, g_y = 1, g_\theta = 0, g_\phi = a\cos\theta$. We want to find $f(x, y, \theta, \phi)$ such that it satisfies $\binom{4}{2} = 6$ conditions as follows:

$$\begin{aligned}
\frac{\partial(fg_x)}{\partial y} &= \frac{\partial(fg_y)}{\partial x} \quad &&\Rightarrow \quad 0 = \frac{\partial f}{\partial x} \\
\frac{\partial(fg_x)}{\partial \theta} &= \frac{\partial(fg_\theta)}{\partial x} \quad &&\Rightarrow \quad 0 = 0 \\
\frac{\partial(fg_x)}{\partial \phi} &= \frac{\partial(fg_\phi)}{\partial x} \quad &&\Rightarrow \quad 0 = a\cos\theta\frac{\partial f}{\partial x} \\
\frac{\partial(fg_y)}{\partial \theta} &= \frac{\partial(fg_\theta)}{\partial y} \quad &&\Rightarrow \quad \frac{\partial f}{\partial \theta} = 0 \qquad (4.5) \\
\frac{\partial(fg_y)}{\partial \phi} &= \frac{\partial(fg_\phi)}{\partial y} \quad &&\Rightarrow \quad \frac{\partial f}{\partial \phi} = a\cos\theta\frac{\partial f}{\partial y} \\
\frac{\partial(fg_\theta)}{\partial \phi} &= \frac{\partial(fg_\phi)}{\partial \theta} \quad &&\Rightarrow \quad 0 = a\cos\theta\frac{\partial f}{\partial \theta} - af\sin\theta \qquad (4.6)
\end{aligned}$$

Using Eq. (4.5) in Eq. (4.6), we conclude that there is no non-trivial integrating factor f unless $\theta = 0$ throughout. Hence Eq. (4.2) cannot be written as an exact differential unless $\theta = 0$ throughout.

If $\theta = 0$, the disk is rolling parallel to y-axis; Eqs. (4.1) and (4.2) can then be integrated and the resulting relation between the coordinates (due to the constraint that the disk is rolling without slipping) will be:

$$\begin{aligned}
y - y_o = -a(\phi - \phi_o), &\qquad \text{where } y_o \text{ and } \phi_o \text{ are constants.} \\
x - x_0 = 0, &\qquad \text{where } x_0 \text{ is a constant.}
\end{aligned}$$

which are holonomic equations of constraint. In this case too, the degrees of freedom of the system further reduce by three (as we now have three more holonomic constraint equation, one each in θ, x and y). Hence we need just one generalized coordinate (a simple choice will be ϕ) to describe the system when $\theta = 0$ throughout.

But there doesn't exist any general integrating factor f such that Eq. (4.2) reduces to exact differential and hence Eq. (4.2) is nonholonomic differential constraint condition.

Problem 5

Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\begin{aligned}
\cos\theta dx + \sin\theta dy &= 0 \\
\sin\theta dx - \cos\theta dy &= \frac{1}{2}a(d\phi + d\phi'),
\end{aligned}$$

(where θ, ϕ, ϕ' have meanings similar to those in the problem of a single vertical disk, and (x, y) are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi'),$$

where C is a constant.

Solution 5

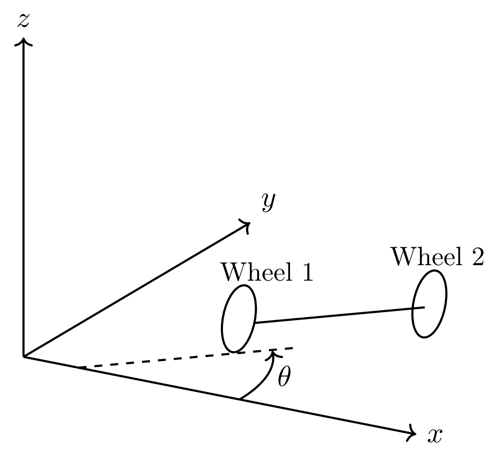

Figure 1: A schematic of the Problem 5

A schematic of the Problem 5 is shown in Figure 1.Two rigid wheels in three dimensions has 12 degrees of freedom. As in the case of a single disk of Problem 4, we have the following holonomic equations of constraints:

$$\begin{aligned}
z_1 &= m, && \text{where } m \text{ is a constant} \\
z_2 &= m, && \text{where } m \text{ is a constant} \\
n_{z_1} &= 0, && \text{where } n_{z_1} \text{ is the component of wheel 1's unit normal vector parallel to } z\text{-axis} \\
n_{z_1} &= 0, && \text{where } n_{z_2} \text{ is the component of wheel 2's unit normal vector parallel to } z\text{-axis}
\end{aligned}$$

Additionaly, the wheels are mounted on an axle of length b, which can be described in the form of holonomic constraint equation as:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = b,$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the centers of wheel 1 and wheel 2 respectively.

Hence the degrees of freedom of the system reduces to seven. We will thus use (x_1, y_1, θ, ϕ) and $(x_2, y_2, \theta, \phi^{'})$ to describe wheel 1 and wheel 2 respectively, where (x_1, y_1) and (x_2, y_2) are the centers of the respective wheels; θ is the angle between x-axis and the common axle; ϕ and $\phi^{'}$ are the angles describing the rotation of the wheels about their respective axis of rotation. Since the wheels are constrained to rolling independently on a plane without slipping, we have the following constraint conditions:

$$\dot{x}_1 = a\dot{\phi}\sin\theta \qquad \Rightarrow \qquad dx_1 = a\sin\theta d\phi \tag{5.1}$$

$$\dot{y}_1 = -a\dot{\phi}\cos\theta \qquad \Rightarrow \qquad dy_1 = -a\cos\theta d\phi \tag{5.2}$$

$$\dot{x}_2 = a\dot{\phi}'\sin\theta \qquad \Rightarrow \qquad dx_2 = a\sin\theta d\phi^{'} \tag{5.3}$$

$$\dot{y}_2 = -a\dot{\phi}'\cos\theta \qquad \Rightarrow \qquad dy_2 = -a\cos\theta d\phi^{'} \tag{5.4}$$

Eqs. (5.1)–(5.4) are nonholomic constraint equations (These are the same equations we proved to be nonholonomic for a single disk in Problem 4; we just have 2 sets of those equations as we now have two wheels in Problem 5).

It is given that (x, y) are the coordinates of a point on the axle midway between the two wheels. Hence we have $(x, y) = \frac{1}{2}(x_1 + x_2, y_1 + y_2)$ and using Eqs. (5.1)–(5.4), we can write dx and dy as:

$$dx = \frac{1}{2}(dx_1 + dx_2) \qquad \Rightarrow \qquad dx = \frac{1}{2}a\sin\theta(d\phi + d\phi^{'}) \tag{5.5}$$

$$dy = \frac{1}{2}(dy_1 + dy_2) \qquad \Rightarrow \qquad dy = -\frac{1}{2}a\cos\theta(d\phi + d\phi^{'}) \tag{5.6}$$

Multiplying Eq. (5.5) by $\cos\theta$ and Eq. (5.6) by $\sin\theta$ and then adding, we get:

$$\boxed{\cos\theta dx + \sin\theta dy = 0} \tag{5.7}$$

Multiplying Eq. (5.5) by $\sin\theta$ and Eq. (5.6) by $\cos\theta$ and then subtracting the latter from the former, we get:

$$\boxed{\sin\theta dx - \cos\theta dy = \frac{1}{2}a(d\phi + d\phi^{'})} \tag{5.8}$$

Performing a similar analysis as done in Problem 5, it is easy to prove that no integrating factor exists for Eqs. (5.7) and (5.8). Hence Eqs. (5.7) and (5.8) are non-holonomic equations of constraint.

From Figure 1, we can write:

$$\begin{aligned}
& x_2 - x_1 = b\cos\theta \\
\Rightarrow \quad & \dot{x}_2 - \dot{x}_1 = -b\dot{\theta}\sin\theta \\
\Rightarrow \quad & a\sin\theta(\dot{\phi}^{'} - \dot{\phi}) = -b\dot{\theta}\sin\theta && \text{(Using Eqs.(5.3) and (5.1))} \\
\Rightarrow \quad & \dot{\theta} = -\frac{a}{b}(\dot{\phi}^{'} - \dot{\phi}) \\
\Rightarrow \quad & \boxed{\theta = C - \frac{a}{b}(\phi^{'} - \phi)} && \text{(C is a constant of integration)} \qquad (5.9)
\end{aligned}$$

Eq. (5.9) is a holonomic constraint equation.

Problem 6

A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

Solution 6

Let the position vector $\mathbf{r}_p$ of the particle with respect to some arbitrary origin O be:

$$\mathbf{r}_p = x\hat{i} + y\hat{j} \tag{6.1}$$

The position vector $\mathbf{r}_x$ of the point on x-axis where the particle's velocity vector is always directed, with respect to the origin O, is given by:

$$\mathbf{r}_x = f(t)\hat{i} \tag{6.2}$$

The position vector $\mathbf{r}$ of the point on x-axis where the particle's velocity is always directed, with respect to the particle, is thus given by:

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_x - \mathbf{r}_p \\ &= (f(t) - x)\hat{i} - y\hat{j} \qquad \text{[Using Eqs. (6.1) and (6.2)]} \end{aligned} \tag{6.3}$$

Differentiating Eq. (6.1) with respect to time, the velocity vector $\mathbf{v}$ of the particle is given by:

$$\mathbf{v} = \dot{x}\hat{i} + \dot{y}\hat{j} \tag{6.4}$$

Note that $\mathbf{v} \parallel \mathbf{r}$. Hence, $\mathbf{v} = \lambda\mathbf{r}$, where λ is a constant. Using Eqs. (6.3) and (6.4), this reduces to the following equation:

$$\frac{\dot{x}}{f(t) - x} = -\frac{\dot{y}}{y} \qquad \Rightarrow \qquad \boxed{ydx + (x - f(t))dy = 0} \tag{6.5}$$

Let us analyze whether Eq. (6.5) is holonomic or not. We want to find an integrating factor $h(x, y)$ on multiplying which Eq. (6.5) reduces to an exact differential. Hence, we want:

$$\begin{aligned} \frac{\partial(hy)}{\partial y} &= \frac{\partial(hx - hf(t)))}{\partial x} \\ \Rightarrow \quad h + y\frac{\partial h}{\partial y} &= h + (x - f(t))\frac{\partial h}{\partial x} \\ \Rightarrow \quad y\frac{\partial x}{\partial y} &= x - f(t) \\ \Rightarrow \quad x &= f(t) \qquad [\because \text{ x is independent of } y] \end{aligned} \tag{6.6}$$

Using Eq. (6.6) in Eq. (6.5), we find that either $dx = 0 \Rightarrow x$ is a constant, which is inconsistent with Eq. (6.6); or $y = 0 \Rightarrow \mathbf{r} = 0$ [Using Eq. (6.3) in Eq. (6.4)] and since $\mathbf{v} = \lambda\mathbf{r}$ (λ is a constant) $\Rightarrow \mathbf{v} = 0 \Rightarrow \dot{x} = 0$ (since $\hat{i}$ and $\hat{j}$ are linearly independent) $\Rightarrow f(t)$ is a constant which is inconsistent since f is given to be a function of t; or if both y and dx are zero, we will encounter the same inconsistency, i.e., $x = f(t)$ is a constant.

Hence, there doesn't exist any integrating function $h(x, y)$ on multiplying which Eq. (6.5) reduces to an exact differential. Therefore, Eq. (6.5) is non-integrable and hence, non-holonomic equation of constraint.

Problem 7

Show that Lagrange's equations in the form of Eqs. (1.53) can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$$

These are sometimes known as the *Nielsen* form of the Lagrange equations.

Remark: By Eqs. (1.53) of the book Classical Mechanics, the author(s) is/are referring to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial \dot{q}_j} = Q_j \tag{7.1}$$

Solution 7

Let $T = T(q, \dot{q}, t)$. Then we have:

$$\dot{T} = \frac{dT}{dt} = \sum_i \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial t} \tag{7.2}$$

Using Eq. (7.2), we can write the following:

$$\begin{aligned}
& \frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial q_i}\dot{q}_i + \frac{\partial T}{\partial q_j} + \frac{\partial^2 T}{\partial \dot{q}_j \partial t} \\
\Rightarrow \quad & \frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial q_i}\dot{q}_i + \frac{\partial^2 T}{\partial \dot{q}_j \partial t} \\
& = \sum_i \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\ddot{q}_i + \sum_i \frac{\partial}{\partial q_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \\
& = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)
\end{aligned} \tag{7.3}$$

Using Eq. (7.3) in Eq. (7.1), we get:

$$\boxed{\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j}$$

which is the required *Nielsen* form of the Lagrange equations.

Problem 8

If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L^{'} = L + \frac{dF(q_1, \ldots, q_n, t)}{dt} \tag{8.1}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

Solution 8

Note that

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t} \tag{8.2}$$

Using Eqs. (8.1) and (8.2), we have:

$$\frac{\partial L^{'}}{\partial q_j} = \frac{\partial L}{\partial q_j} + \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 F}{\partial q_j \partial t} \tag{8.3}$$

Again using Eq. (8.1) and (8.2), we have:

$$\begin{aligned} \frac{\partial L'}{\partial \dot{q}_j} &= \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial F}{\partial q_j} \\ \Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right) \end{aligned} \tag{8.4}$$

Rewriting Eq. (8.4) after expanding the second term on its right hand side using chain rule, we get:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_i \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_i + \frac{\partial^2 F}{\partial t \partial q_j} \\ \Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 F}{\partial q_j \partial t} \end{aligned} \tag{8.5}$$

Subtracting Eq. (8.3) from Eq. (8.5), we get:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 F}{\partial q_j \partial t} - \frac{\partial L}{\partial q_j} - \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i - \frac{\partial^2 F}{\partial q_j \partial t} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad (\text{ or } Q_j \text{ if non-conservative forces are present}) \end{aligned}$$

Hence, if L satisfies Lagrange's equations then L', given by Eq. (8.1), also satisfies Lagrange's equations.

Problem 9

The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\begin{aligned} &\mathbf{A} \longrightarrow \mathbf{A} + \frac{1}{c}\boldsymbol{\nabla}\psi(\mathbf{r}, t), \\ &\phi \longrightarrow \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}, \end{aligned}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of the particle moving in the electromagnetic field? Is the motion affected?

Solution 9

The initial Lagrangian is given by:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi + q\mathbf{A}.\mathbf{v} \tag{9.1}$$

On performing the given gauge transformation, the Lagrangian transforms to the following :

$$\begin{aligned}
L' &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\phi' + q\mathbf{A}'.\mathbf{v} \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\phi + \frac{q}{c}\frac{\partial\psi}{\partial t} + q\mathbf{A}.\mathbf{v} + \frac{q}{c}\boldsymbol{\nabla}\psi(\mathbf{r},t).\mathbf{v} \\
&= L + \frac{q}{c}\left(\frac{\partial\psi}{\partial x}\dot{x} + \frac{\partial\psi}{\partial y}\dot{y} + \frac{\partial\psi}{\partial z}\dot{z} + \frac{\partial\psi}{\partial t}\right) \qquad \text{[Using Eq. (9.1)]} \\
&= \boxed{L + \frac{d}{dt}\left(\frac{q}{c}\psi(\mathbf{r},t)\right)} \qquad (9.2)
\end{aligned}$$

Eq. (9.2) is of the same form as Eq. (8.1). Using the result of the Problem 8, the motion is thus not affected by performing the given gauge transformation.

Problem 10

Let $q_1,\dots,q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom, with a Lagrangian $L(q,\dot{q},t)$. Suppose we transform to another set of independent coordinates $s_1,\dots,s_n$ by means of transformation equations

$$q_i = q_i(s_1,\dots,s_n,t), \qquad i=1,\dots,n.$$

(Such a transformation is called a point transformation.) Show that if the Lagrangian function is expressed as a function of $s_j, \dot{s}_j$, and t through the equations of transformation, then L satisfies Lagrange's equations with respect to the s coordinates:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0.$$

In other words, the form of Lagrange's equations is invariant under a point transformation.

Solution 10

We want to show that if $L(q,\dot{q},t)$ satisfies Lagrange's equations then $L(s,\dot{s},t) = L(q(s,t),\dot{q}_j(s,\dot{s},t),t)$ also satisfies Lagrange's equations. Note that:

$$\dot{q}_i = \sum_k \frac{\partial q_i}{\partial s_k}\dot{s}_k + \frac{\partial q_i}{\partial t} \qquad (10.1)$$

$$\Rightarrow \qquad \frac{\partial \dot{q}_i}{\partial \dot{s}_j} = \frac{\partial q_i}{\partial s_j} \qquad (10.2)$$

Now, using chain rule, we have the following:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{s}_j} &= \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{s}_j} \\
\Rightarrow \qquad \frac{\partial L}{\partial \dot{s}_j} &= \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \dot{s}_j} \qquad \text{[Using Eq. (10.2)]} \\
\Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) &= \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \dot{s}_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{\partial q_i}{\partial \dot{s}_j}\right)\right] \qquad (10.3)
\end{aligned}$$

Again using chain rule, we have the following:

$$\frac{\partial L}{\partial s_j} = \sum_i \left[\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial s_j} \right] \tag{10.4}$$

Using Eq. (10.1), we have the following:

$$\frac{\partial \dot{q}_i}{\partial s_j} = \sum_k \frac{\partial^2 \dot{q}_i}{\partial s_j \partial s_k} \dot{s}_k + \frac{\partial^2 \dot{q}_i}{\partial s_j \partial t} = \sum_k \frac{\partial}{\partial s_k} \left(\frac{\partial \dot{q}_i}{\partial s_j} \right) \dot{s}_k + \frac{\partial}{\partial t} \left(\frac{\partial \dot{q}_i}{\partial s_j} \right) = \frac{d}{dt} \left(\frac{\partial \dot{q}_i}{\partial s_j} \right) \tag{10.5}$$

Using Eq. (10.5) in Eq. (10.4), we get:

$$\frac{\partial L}{\partial s_j} = \sum_i \left[\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \left(\frac{\partial q_i}{\partial s_j} \right) \right] \tag{10.6}$$

Subtracting Eq. (10.6) from Eq. (10.3), we get:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}_j} \right) - \frac{\partial L}{\partial s_j} &= \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial \dot{q}_i} \right] \frac{\partial q_i}{\partial s_j} \\ &= 0 \qquad [\because L(q, \dot{q}, t) \text{ satisfies Lagrange's equations}] \end{aligned}$$

Thus $L(s, \dot{s}, t)$ also satisfies Lagrange's equations. Hence, the form of Lagrange's equations is invariant under a point transformation.

Problem 11

Check whether the force $\vec{F} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$ is conservative or not.

Solution 11

To check whether the force $\vec{F} = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}$ is conservative or not, we compute the curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & zx & xy \end{vmatrix} = (x - x)\hat{\imath} - (y - y)\hat{\jmath} + (z - z)\hat{k} = \vec{0}$$

Since $\nabla \times \vec{F} = \vec{0}$ and the domain ($\mathbb{R}^3$) is simply connected, $\vec{F}$ is conservative.

Problem 12

Compute the orbital period and orbital angular velocity of a satellite revolving around the Earth at an altitude of 720 km. [Given: radius of Earth R=6000 km and g=9.83 m/s^2.]

Solution 12

We will assume a circular orbit. At the surface of the Earth, Newton's law of Gravitation states:

$$\begin{aligned} \frac{GMm}{R^2} &= mg \\ \Rightarrow \quad GM &= gR^2 \\ \Rightarrow \quad GM &= 9.83 \times (6 \times 10^6)^2 \text{ m}^3/\text{s}^2 \\ \Rightarrow \quad GM &= 3.54 \times 10^{14} \text{ m}^3/\text{s}^2 \end{aligned} \tag{12.1}$$

At a distance $r = R + h = 6.72 \times 10^6$ m from the center of mass of the earth, Newton's law of Gravitation states:

$$\begin{aligned} \frac{GMm}{r^2} &= mr\omega^2 \\ \Rightarrow \quad \omega = \sqrt{\frac{GM}{r^3}} &= \sqrt{\frac{58.98 \times 10^6}{(6.72 \times 10^6)^3}} \text{ rad/s} && \text{[Using Eq. (12.1)]} \\ &\approx \boxed{1.08 \times 10^{-3} \text{ rad/s}} \end{aligned} \tag{12.2}$$

We know:

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2 \times 3.14}{1.08 \times 10^{-3}} \text{ s} && \text{[Using Eq. (12.2)]} \\ &\approx \boxed{5814.81 \text{ s}} \end{aligned}$$

Problem 13

Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmosheric friction, is

$$m\frac{dv}{dt} = -v^{'}\frac{dm}{dt} - mg,$$

where m is the mass of the rocket and $v^{'}$ is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with $v^{'}$ equal to 2.1 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

Solution 13

We choose Earth as our frame of reference. At time t, let the mass of the rocket be m and its velocity be $v\hat{k}$. At time $t + dt$, the mass of the rocket reduces to $m - dm$ and its velocity increases to $(v + dv)\hat{k}$ after expelling gases with mass dm and with velocity $-v_e\hat{k}$ with respect to the Earth.

The change in momentum dp in time interval dt is thus:

$$\begin{aligned} dp.\,\hat{k} &= (m - dm)(v + dv)\hat{k} - dm.\,v_e\hat{k} - mv\hat{k} \\ &= [mdv - dm(v_e + v)]\hat{k} \\ \Rightarrow \quad dp.\,\hat{k} &= [mdv + v^{'}dm]\hat{k} \qquad [\because v^{'}\hat{k} = -(v_e + v)\hat{k}] \end{aligned} \tag{13.1}$$

The rate of change of momentum is:

$$\frac{dp}{dt}\hat{k} = m\frac{dv}{dt}\hat{k} + v^{'}\frac{dm}{dt}\hat{k} \qquad \text{[Using Eq. (13.1)]} \tag{13.2}$$

Using Newton's second law, we have:

$$\begin{aligned} \frac{dp}{dt}\hat{k} &= \mathbf{F} = -mg\hat{k} \\ \Rightarrow \quad & \boxed{m\frac{dv}{dt} = -v^{'}\frac{dm}{dt} - mg} \qquad \text{[Using Eq. (13.2)]} \end{aligned} \tag{13.3}$$

Eq. (13.3) can be rewritten using $\dot{m} = \dfrac{dm}{dt}$ as:

$$dv = -\frac{v^{'}}{m}dm - gdt = -\frac{v^{'}}{m}dm - \frac{g}{\dot{m}}dm \tag{13.4}$$

Let m_0 and v_0 be the initial mass and the initial velocity of the rocket respectively. Assuming $\dot{m} = -\frac{1}{60}m_0$, $v^{'} = 2.1$ km/s $= 2100$ m/s and $g = 9.83$ m/s^2 to be constants and integrating Eq. (13.4), we get:

$$\begin{aligned} v - v_0 &= v^{'}\ln\left|\frac{m_0}{m}\right| + \frac{g}{\dot{m}}(m_0 - m) \\ \Rightarrow \quad v &= 2100 \times \ln\left|\frac{m_0}{m}\right| + 60 \times 9.83\left(\frac{m}{m_0} - 1\right) \qquad [\because v_0 = 0] \\ \Rightarrow \quad v &= 2100 \times \ln\left|\frac{m_0}{m}\right| + 589.8\left(\frac{m}{m_0} - 1\right) \end{aligned} \tag{13.5}$$

The escape velocity from the surface of Earth is $v = 11.2$ km $= 11200$ m/s. Let $r = \frac{m_0}{m}$. Substituting these in Eq. (13.5), we get:

$$\begin{aligned} 11200 &= 2100 \times \ln|r| + 589.8\left(\frac{1}{r} - 1\right) \\ \Rightarrow \quad & \boxed{\frac{589.8}{r} - 2100\ln|r| - 11789.8 = 0} \end{aligned} \tag{13.6}$$

Eq. (13.6) is a transcendental equation and is to be numerically solved. I will implement the *Newton-Raphson* root-finding algorithm in Python to solve Eq. (13.6) and will use matplotlib module to depict the result graphically.

```
# importing modules through alias
import numpy as np
import matplotlib.pyplot as plt
```

```
# Defining the function
def f(r):
    return 589.8/r + 2100*np.log(r) - 11789.8

# Defining the function's derivative
def f_prime(r):
    return -589.8/(r**2) + 2100/r

# Newton-Raphson implementation
def newton_raphson(r0, tol, max_iter):
    r = r0
    for i in range(max_iter):
        fr = f(r)
        fpr = f_prime(r)
        if fpr == 0:
            raise ValueError("Derivative is zero. No convergence.")
        r_new = r - fr/fpr
        if abs(r_new - r) < tol:
            return r_new, i+1
        r = r_new
    raise ValueError("Did not converge within max iterations")

# Initial guess
r0 = 300
solution, iterations = newton_raphson(r0,10**(-3),100)
print(f"Newton-Raphson solution: r is approximately {solution:.2f} (in {
    iterations} iterations)")

# Creating array of r values and corresponding f values
r_vals = np.linspace(1, 500, 1000)
f_vals = f(r_vals)

# Plotting the function
plt.figure(figsize=(10, 6))
plt.plot(r_vals, f_vals, label=r'$f(r)$', color='blue')
plt.scatter(solution,0, color='red',s=10,zorder=3, label=fr'Solution: r $\
    approx$ {solution:.2f}')
plt.xlabel('r')
plt.ylabel('f(r)')
plt.xlim(1,500)
plt.title('Graphical depiction of the root')
plt.grid()
plt.legend()
plt.show()
```

From Figure 2, we have $r = \frac{m_0}{m} \approx 274.01$.

The fuel-to-empty rocket weight ratio to achieve the escape velocity is greater than or equal to $\dfrac{m_0 - m}{m} = r - 1 \approx 273.01$ (almost 300!).

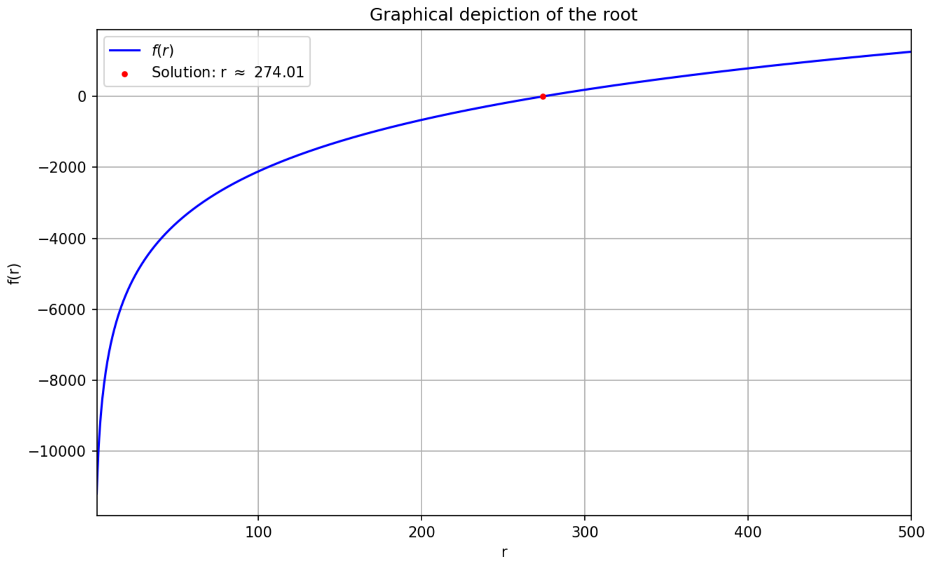

Figure 2: Graphical depiction of the root of Eq. (13.6)

Problem 14

Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

Solution 14

Two particles in three dimensions have six degrees of freedom. The particles connected by a weightless rod of length l reduces one degree of freedom and the center of rod (center of mass of the system) constrained to move in a circle of radius a further reduces two degrees of freedom. Hence the system has three degrees of freedom. We will use (η, θ, ϕ) as our generalized coordinates, whose geometric interpretations are detailed below.

The kinetic energy of the system will have two components - the kinetic energy of the center of mass and summation of the kinetic energies of particles with respect to the center of mass. Let η be the angle representing the center of mass' position on circle which is centered at the origin. The position vector $\mathbf{R}$, and hence the of the velocity vector $\mathbf{v}$, of the center of mass of the system with respect to the origin is given as:

$$\begin{aligned} \mathbf{R} &= a(\cos\eta\hat{i} + \sin\eta\hat{j}) \\ \Rightarrow \qquad \mathbf{v} =& \dot{\mathbf{R}} = a\dot{\eta}(-\sin\eta\hat{i} + \cos\eta\hat{j}) \end{aligned} \tag{14.1}$$

Using Eq. (14.1), the kinetic energy T_1 of the center of mass of the system is:

$$T_1 = \frac{1}{2}(2m)\mathbf{v}\cdot\mathbf{v} = ma^2\dot{\eta}^2 \tag{14.2}$$

Let us use (θ, ϕ), as in spherical coordinates, to describe the orientation of the particles with respect to the rectangular coordinate system having origin at the center of mass of the system such that one particle's coordinates are reflection under origin of the other particle's coordinates, i.e., $\mathbf{r}_2' = -\mathbf{r}_1'$, where $\mathbf{r}_1^{'}$ and $\mathbf{r}_2^{'}$ are the position vectors of the particles with respect to the center of mass. Let θ, ϕ reprersent the first particle. Then by our choice of cordinate system, the second particle will be represented by $\pi - \theta$ and $\pi + \phi$. We thus have:

$$\mathbf{r}_1^{'} = \frac{l}{2}(\sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k}) \tag{14.3}$$

$$\mathbf{r}_2^{'} = -\frac{l}{2}(\sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k}) \tag{14.4}$$

Using Eqs. (14.3) and (14.4), the corresponding velocity vectors $\mathbf{v}_1^{'}$ and $\mathbf{v}_2^{'}$ with respect to the center of mass are:

$$\mathbf{v}_1^{'} = \dot{\mathbf{r}}_1^{'} = \frac{l}{2}[(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\hat{\imath} + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\hat{\jmath} - \dot{\theta}\sin\theta\hat{k}] \tag{14.5}$$

$$\mathbf{v}_2^{'} = \dot{\mathbf{r}}_2^{'} = -\frac{l}{2}[(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\hat{\imath} + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\hat{\jmath} - \dot{\theta}\sin\theta\hat{k}] \tag{14.6}$$

Using Eqs. (14.5) and (14.6), the summation T_2 of the kinetic energies of particles about the center of mass is given as:

$$\begin{aligned}
T_2 &= \frac{1}{2}m\mathbf{v}_1^{'}\cdot\mathbf{v}_1^{'} + \frac{1}{2}m\mathbf{v}_2^{'}\cdot\mathbf{v}_2^{'} \\
&= \frac{1}{2}m.\,2\frac{l^2}{4}(\dot{\theta}^2\cos^2\theta\cos^2\phi + \dot{\phi}^2\sin^2\theta\sin^2\phi + \dot{\theta}^2\cos^2\theta\sin^2\phi + \dot{\phi}^2\sin^2\theta\cos^2\phi + \dot{\theta}^2\sin^2\theta) \\
&= \frac{ml^2}{4}[\dot{\theta}^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + \dot{\phi}^2\sin^2\theta(\sin^2\phi + \cos^2\phi) + \dot{\theta}^2\sin^2\theta] \\
&= \frac{ml^2}{4}[\dot{\theta}^2(\cos^2\theta + \sin^2\theta) + \dot{\phi}^2\sin^2\theta] \\
&= \frac{ml^2}{4}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)
\end{aligned} \tag{14.7}$$

Using Eqs. (14.2) and (14.7), the kinetic energy T of the system in generalized coordinates (η, θ, ϕ) is:

$$\begin{aligned}
T &= T_1 + T_2 \\
&= \boxed{ma^2\dot{\eta}^2 + \frac{ml^2}{4}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)}
\end{aligned}$$

Problem 15

A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \boldsymbol{\sigma}.\mathbf{L},$$

where $\mathbf{r}$ is the radius vector from a fixed point, $\mathbf{L}$ is the angular momentum about that point, and $\boldsymbol{\sigma}$ is a vector fixed in space.

(a) Find the components of the force on the particle in both Cartesian and spherical polar coordinates, on the basis of Eq.(1.58).

(b) Show that the components in the two coordinate systems are related to each other as in Eq. (1.49).

(c) Obtain the equations of motion in spherical polar coordinates.

Remark: By Eqs. (1.49) and (1.58) of the book Classical Mechanics, the author(s) is/are referring to

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{q_j} \qquad \text{and}$$
$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \qquad \text{respectively}$$

Solution 15: *This solution is still in progress. I attempted the following approach:*

The given generalized potential U can be written as:

$$U(\mathbf{r}, \mathbf{v}) = V(r) + m\boldsymbol{\sigma} \cdot (\mathbf{r} \times \mathbf{v}) = V(r) + m\mathbf{r} \cdot (\mathbf{v} \times \boldsymbol{\sigma}) = V(r) + m\mathbf{v} \cdot (\boldsymbol{\sigma} \times \mathbf{r}) \tag{15.1}$$

(a) First we find components of force in cartesian coordinates as follows:

$$\begin{aligned}
Q_i &= \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i} \\
&= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}_i}(m\boldsymbol{\sigma} \cdot (\mathbf{r} \times \mathbf{v}))\right] - V'(r)\frac{x_i}{r} - m\frac{\partial}{\partial x_i}(\mathbf{v} \cdot (\boldsymbol{\sigma} \times \mathbf{r})) \qquad \text{(Using Eq. (15.1))} \\
&= \frac{d}{dt}[m(\boldsymbol{\sigma} \times \mathbf{r}_i)] - V'(r)\frac{x_i}{r} - m(\mathbf{v} \times \boldsymbol{\sigma})_i \\
&= m(\boldsymbol{\sigma} \times \mathbf{v})_i - V'(r)\frac{x_i}{r} - m(\mathbf{v} \times \boldsymbol{\sigma})_i \\
&= -V'(r)\frac{x_i}{r} + m((\boldsymbol{\sigma} \times \mathbf{v}) - (\mathbf{v} \times \boldsymbol{\sigma}))_i \\
&= \boxed{2m(\boldsymbol{\sigma} \times \mathbf{v})_i - V'(r)\frac{x_i}{r}}
\end{aligned} \tag{15.2}$$

Problem 16

A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right),$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. (The expression for F represents the force between two charges in Weber's electrodynamics.)

Solution 16

The force is given by:

$$\begin{aligned} F &= \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right) \\ &= \frac{1}{r^2} - \frac{\dot{r}^2}{r^2c^2} + \frac{2\ddot{r}}{rc^2} \\ &= \frac{1}{r^2} + \frac{\dot{r}^2}{r^2c^2} + \frac{2\ddot{r}}{rc^2} - \frac{2\dot{r}^2}{r^2c^2} \\ &= -\frac{\partial}{\partial r}\left(\frac{1}{r} + \frac{\dot{r}^2}{rc^2}\right) + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{r}}\left(\frac{1}{r} + \frac{\dot{r}^2}{rc^2}\right)\right] \\ &= -\frac{\partial U}{\partial r} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) \quad , \text{ where } U = \left(\frac{1}{r} + \frac{\dot{r}^2}{rc^2}\right) \end{aligned} \tag{16.1}$$

From Eq. (16.1), we conclude that the generalized potential U is given by:

$$\boxed{U(r,\dot{r}) = \left(\frac{1}{r} + \frac{\dot{r}^2}{rc^2}\right)} \tag{16.2}$$

Since the motion is restricted to a plane, we have in polar coordinates (r, θ):

$$\begin{aligned} & d\mathbf{r} = \hat{r}dr + \hat{\theta}rd\theta \\ \Rightarrow \quad & \mathbf{v} = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} \\ \Rightarrow \quad & v^2 = \mathbf{v}\cdot\mathbf{v} = \dot{r}^2 + r^2\dot{\theta}^2 \end{aligned} \tag{16.3}$$

Using Eq. (16.3), the kinetic energy T of the particle will be:

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \end{aligned} \tag{16.4}$$

Using Eqs. (16.3) and (16.4), the Lagrangian L of the system can be written as:

$$\begin{aligned} L &= T - U \\ &= \boxed{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \left(\frac{1}{r} + \frac{\dot{r}^2}{rc^2}\right)} \end{aligned}$$

Problem 17

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 Mev/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. (The MeV, million electron volt, is a unit of energy used in modern physics, equal to 1.60×10^{-13} J. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34×10^{-22} kg·m/s.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90×10^{-25} kg what is its kinetic energy, in electron volts?

Solution 17

Let us choose 2 dimensional cartesian coordinate system for this problem. Let the nucleus be at origin. It is given that the nucleus is at rest. Hence, the initial momentum is:

$$\mathbf{p}_{\text{initial}} = [0\hat{\imath} + 0\hat{\jmath}] \text{ MeV}/c \tag{17.1}$$

Emitted in a radioactive decay, let the electron be travelling along positive x-axis and let the neutrino be travelling along positive y-axis. When the residual nucleus recoils, let its linear momentum be $\mathbf{p}_{\text{recoil}} = p_x\hat{\imath} + p_y\hat{\jmath}$. Hence, using the given data, the final momentum is:

$$\mathbf{p}_{\text{final}} = [(1.73 + p_x)\hat{\imath} + (1.00 + p_y)\hat{\jmath}] \text{ MeV}/c \tag{17.2}$$

Since there is no external force, all components of the linear momentum are conserved. Hence, using Eq. (17.1) and Eq. (17.2), we have:

$$\mathbf{p}_{\text{recoil}} = [-1.73\hat{\imath} - 1.00\hat{\jmath}] \text{ MeV}/c \tag{17.3}$$

Using Eq. (17.3), the residual nucleus recoils at an angle $\pi + \theta$ with respect to the positive direction of x-axis, where:

$$\pi + \theta = \left[180 + \arctan\left(\frac{1.00}{1.73}\right)\right]^\circ \approx \boxed{180.33^\circ}$$

Using Eq. (17.3), we can write:

$$\begin{aligned}
\|\mathbf{p}_{\text{recoil}}\| &= \sqrt{\mathbf{p}_{\text{recoil}} \cdot \mathbf{p}_{\text{recoil}}} \\
&= \sqrt{(-1.73)^2 + (-1.00)^2} \text{ MeV}/c \\
&\approx \boxed{2.00 \text{ MeV}/c}
\end{aligned}$$

Using given data and Eq. (17.3), we have:

$$\begin{aligned}
T &= \frac{\mathbf{p}_{\text{recoil}} \cdot \mathbf{p}_{\text{recoil}}}{2m} \\
&= \frac{(-1.73 \times 5.34 \times 10^{-22})^2 + (-1.00^2 \times 5.34 \times 10^{-22})^2}{2 \times 3.90 \times 10^{-25}} \text{ J} \\
&\approx 1.46 \times 10^{-18} \text{ J} \\
&= \frac{1.46 \times 10^{-18}}{1.60 \times 10^{-19}} \text{ eV} \\
&\approx \boxed{9.12 \text{ eV}}
\end{aligned}$$

Problem 18

A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2),$$

where a,b and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0, c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq. (1.56) is related to L' by a point transformation (cf. Derivation 10). What is the significance of the condition on the value of $b^2 - ac$?

Remark: By Eq. (1.56) of the book Classical Mechanics, the author(s) is/are referring to

$$L = T - V$$

and by Derivation 10, the author(s) is/are referring to Problem 10.

Solution 18

Using the given Lagrangian $L^{'}$ in Euler-Lagrange equations, we get the following equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L^{'}}{\partial \dot{x}}\right) - \frac{\partial L^{'}}{\partial x} = 0 \qquad \Rightarrow \qquad \boxed{m(a\ddot{x} + b\ddot{y}) = -K(ax + by)} \tag{18.1}$$

$$\frac{d}{dt}\left(\frac{\partial L^{'}}{\partial \dot{y}}\right) - \frac{\partial L^{'}}{\partial y} = 0 \qquad \Rightarrow \qquad \boxed{m(b\ddot{x} + c\ddot{y}) = -K(bx + cy)} \tag{18.2}$$

If $a = 0 = c$ in Eqs. (18.1) and (18.2), we get:

$$m\ddot{x} = -Kx \qquad ; \qquad m\ddot{y} = -Ky \tag{18.3}$$

If $b = 0, c = -a$ in Eqs. (18.1) and (18.2), we get:

$$m\ddot{x} = -Kx \qquad ; \qquad m\ddot{y} = -Ky \tag{18.4}$$

From Eqs. (18.3) and (18.4), we conclude that both sets of condition gives a Lagrangian (and hence equation of motion) which is associated to the physical system of harmonic oscillation of a particle of mass m in two dimensions.

We perform the following linear point transformation:

$$u = ax + by \qquad ; \qquad v = bx + cy \tag{18.5}$$

Differentiating Eq. (18.5) twice, we get:

$$\dot{u} = a\dot{x} + b\dot{y} \qquad ; \qquad \dot{v} = b\dot{x} + c\dot{y} \tag{18.6}$$

Substituting Eq. (18.6) in Eqs. (18.1) and (18.2), we get:

$$m\ddot{u} = -Ku \qquad ; \qquad m\ddot{v} = -Kv$$

which are the standard equations for two uncoupled identical simple harmonic oscillators. Hence in (u, v) coordinates the Lagrangian can be written in the usual form

$$\boxed{L(u, v, \dot{u}, \dot{v}) = \frac{m}{2}\left(\dot{u}^2 + \dot{v}^2\right) - \frac{K}{2}\left(u^2 + v^2\right)}$$

which is precisely $L = T - V$.

The point transformation $(x, y) \mapsto (u, v)$ could be written in the matrix form as: $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
This point transformation is invertible if and only if

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = ac - b^2 \neq 0.$$

Thus the condition $b^2 - ac \neq 0$ ensures that the change of variables is a valid point transformation and the system truly has two independent degrees of freedom (two oscillators). If $b^2 - ac = 0$, the transformation becomes singular, one linear combination of coordinates disappears, and the system effectively reduces to a one-dimensional oscillator.

Problem 19

Obtain the Lagrange equations of motion for a spherical pendulum, i.e., a mass point suspended by a rigid weightless rod.

Solution 19

We will use spherical polar coordinates (r, θ, ϕ) for solving this problem. A schematic of the problem is shown in Figure 3.

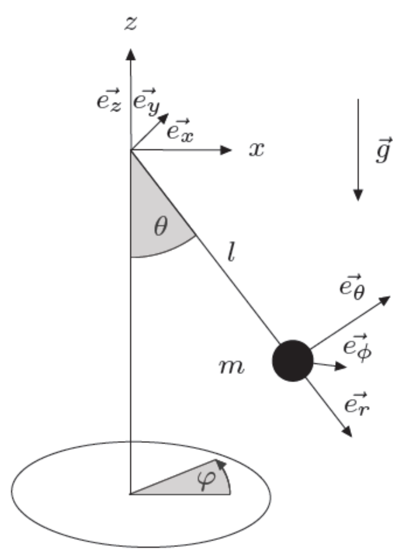

Figure 3: Spherical Pendulum

Since the rod is rigid, we have a holonomic constraint $r = l$, thus reducing the degree of freedom of the system to two. The system can be described by (θ, ϕ). In spherical polar coordinates, we have:

$$\begin{aligned} & d\mathbf{r} = \hat{\mathbf{e}}_{\mathbf{r}} dr + \hat{\mathbf{e}}_{\boldsymbol{\theta}} r d\theta + \hat{\mathbf{e}}_{\boldsymbol{\phi}} r \sin\theta d\phi \\ \Rightarrow \quad & \mathbf{v} = \hat{\mathbf{e}}_{\mathbf{r}} \dot{r} + \hat{\mathbf{e}}_{\boldsymbol{\theta}} r\dot{\theta} + \hat{\mathbf{e}}_{\boldsymbol{\phi}} r \sin\theta \dot{\phi} \\ \Rightarrow \quad & v^2 = \mathbf{v} \cdot \mathbf{v} = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2 \end{aligned} \tag{19.1}$$

The kinetic energy T of the system is given by:

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) && \text{(Using Eq. (19.1))} \\ &= \frac{1}{2}(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2) && \text{(Using } r = l \text{ is a constant)} \end{aligned} \tag{19.2}$$

Using the point of suspension as the reference for potential energy, we have:

$$V = -mgl\cos\theta \tag{19.3}$$

Using Eqs. (19.2) and (19.3), the Lagrangian L of the system can be written as:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2) + mgl\cos\theta \end{aligned} \tag{19.4}$$

The equations of motion are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \qquad \text{and} \tag{19.5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \tag{19.6}$$

Solving Eqs. (19.5) and (19.6) using Eq. (19.4) gives us the following two equations of motion respectively:

$$\boxed{l\ddot{\theta} - \frac{l}{2}\sin 2\theta\dot{\phi}^2 + g\sin\theta = 0} \qquad \text{and} \qquad \boxed{\sin^2\theta\dot{\phi} = \text{constant}}$$

Problem 20

A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x),$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

Solution 20

The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \tag{20.1}$$

Substituting the given Lagrangian L in Eq. (20.1), we get:

$$\begin{aligned} & m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) - m\dot{x}^2V^{'}(x) + 2V(x)V^{'}(x) = 0 \\ \Rightarrow \quad & \boxed{[m^2\dot{x}^2 + 2mV(x)]\ddot{x} = [m\dot{x}^2 - 2V(x)]V^{'}(x)} \end{aligned} \tag{20.2}$$

From Eq. (20.2), we can describe the physical nature of the system as follows:

1. The effective mass depends on both $\dot{x}$ and x: $m_{\text{eff}} = m^2\dot{x}^2 + 2mV(x)$.
2. The effective force is also nonlinear: $F_{\text{eff}} = [m\dot{x}^2 - 2V(x)]V'(x)$.
3. The motion is not simple harmonic; it involves nonlinear oscillations depending on both velocity and position.

Problem 21

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until m_1 reaches the hole.)

Solution 21

Point mass m_1 lies on the table which is a two dimensional surface and hence m_1 can be described by polar coordinates (r, θ) with the origin of coordinate system being at the hole on the table. Point mass m_2 can move about in a vertical line passing through the hole and downward to the hole. We can describe m_2 by y which corresponds to its distance from the hole. Since the string is inextensible (of length l), we have a holonomic constraint given by $r + y = l$. Thus the system has two degrees of freedom and can be described by (r, θ).

The kinetic energy T of the system is:

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\ &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{y}^2 \\ &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 \qquad (\because \dot{y} = -\dot{r}) \end{aligned} \tag{21.1}$$

Let us choose the hole on the table as our reference for potential energy. The potential energy V of the system is:

$$\begin{aligned} V &= V_1 + V_2 \\ &= 0 - m_2 g y \\ &= -m_2 g(l - r) \end{aligned} \tag{21.2}$$

Using Eqs. (21.1) and (21.2), the Lagrangian $\mathcal{L}$ for the system can be written as:

$$\begin{aligned} \mathcal{L} &= T - V \\ &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 + m_2 g(l - r) \end{aligned} \tag{21.3}$$

Using Eq. (21.3), the Euler-Lagrange Equation can be solved for our generalized coordinates as follows:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \quad \Rightarrow \quad \boxed{(m_1 + m_2)\ddot{r} = m_1 r\dot{\theta}^2 - m_2 g r} \quad \text{and} \tag{21.4}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad \Rightarrow \quad \boxed{m_1 r^2\dot{\theta} = \text{constant} = L} \quad (L \equiv \text{angular momentum}) \tag{21.5}$$

Eqs. (21.4) and (21.5) are the equations of motion of the system. Eq. (21.5) says that angular momentum is conserved. Using Eq. (21.5) in Eq. (21.4), we get:

$$\boxed{(m_1 + m_2)\ddot{r} = \frac{L^2}{m_1 r^3} - m_2 g r} \tag{21.6}$$

Thus the problem is reduced to a second order differential equation given by Eq. (21.6). Multiplying Eq. (21.6) by $\dot{r}$, it can be written as:

$$\begin{aligned} (m_1 + m_2)\ddot{r}\dot{r} &= \frac{L^2\dot{r}}{m_1 r^3} - m_2 g r\dot{r} \\ \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}(m_1 + m_2)\dot{r}^2\right] &= \frac{d}{dt}\left(-\frac{L^2}{2m_1 r^2}\right) + \frac{d}{dt}\left(-\frac{1}{2}m_2 g r^2\right) \end{aligned} \tag{21.7}$$

Upon integrating Eq. (21.7), we get the first integral of Eq. (21.6) as follows:

$$\boxed{\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{L^2}{2m_1 r^2} + \frac{1}{2}m_2 g r^2 = \text{constant} = E} \quad (E \equiv \text{total energy}) \tag{21.8}$$

Eq. (21.8) is energy conservation equation with effective potential energy $V_{\text{eff}}(r)$ being:

$$\boxed{V_{\text{eff}}(r) = \frac{L^2}{2m_1 r^2} + \frac{1}{2}m_2 g r^2}$$

Problem 22

Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig 1.4, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

Remark: By Fig. 1.4 of the book Classical Mechanics, the author(s) is/are referring to

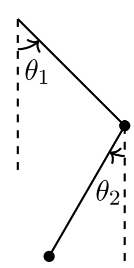

Figure 4: Double pendulum.

Solution 22

Double pendulum in two dimensional vertical plane has four degrees of freedom (two for each mass). Let the origin of our cartesian coorfinate system be at the point of suspension of first mass. Since we have two holonomic constraints given by $\sqrt{x_1^2 + y_1^2} = l_1$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = l_2$, the degrees of freedom of the system reduces to two. We can describe the system in terms of the generalized coordinates θ_1 and θ_2.

The kinetic energy T of the system is given by:

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \end{aligned} \tag{22.1}$$

Now we compute the velocity components for both the masses as follows:

$$x_1 = l_1 \sin\theta_1 \quad \Rightarrow \quad \dot{x}_1 = \dot{\theta}_1 l_1 \cos\theta_1 \tag{22.2}$$

$$y_1 = l_1 \cos\theta_1 \quad \Rightarrow \quad \dot{y}_1 = -\dot{\theta}_1 l_1 \sin\theta_1 \tag{22.3}$$

$$x_2 = l_1 \sin\theta_1 - l_2 \sin\theta_2 \quad \Rightarrow \quad \dot{x}_2 = \dot{\theta}_1 l_1 \cos\theta_1 - \dot{\theta}_2 l_2 \cos\theta_2 \tag{22.4}$$

$$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \quad \Rightarrow \quad \dot{y}_2 = -(\dot{\theta}_1 l_1 \sin\theta_1 + \dot{\theta}_2 l_2 \sin\theta_2) \tag{22.5}$$

Substituting Eqs. (22.2)-(22.5) in Eq. (22.1), we get:

$$T = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 - 2\dot{\theta}_1\dot{\theta}_2 l_1 l_2 \cos(\theta_1 + \theta_2)] \tag{22.6}$$

Let us choose the point of suspension of the first mass as our reference for potential energy. The potential energy V of the system is given by:

$$\begin{aligned} V &= V_1 + V_1 \\ &= -m_1 g y_1 - m_2 g y_2 \\ &= -[m_1 g l_1 \cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)] \end{aligned} \tag{22.7}$$

Using Eqs. (22.5) and (22.6), the Lagrangian L of the system can be written as:

$$\begin{aligned} L &= T - V \\ &= \boxed{\frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 - 2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\cos(\theta_1+\theta_2)] + m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)} \end{aligned} \tag{22.8}$$

The two equations of motion of the system are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 \qquad \text{and} \tag{22.9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0 \tag{22.10}$$

Using Eq. (22.8) to solve Eqs. (22.9) and (22.10), we get the following two equations of motion respectively:

$$\boxed{[m_1 + m_2](l_1^2\ddot{\theta}_1 + g l_1\sin\theta_1) = m_2 l_1 l_2[\ddot{\theta}_2\cos(\theta_1+\theta_2) - \dot{\theta}_2^2\sin(\theta_1+\theta_2)]} \qquad \text{and}$$

$$\boxed{m_2(l_2^2\ddot{\theta}_2 + g l_2\sin\theta_2) = m_2 l_1 l_2[\ddot{\theta}_1\cos(\theta_1+\theta_2) - \dot{\theta}_1^2\sin(\theta_1+\theta_2)]}$$

Problem 23

Two masses 2kg and 3kg, respectively, are tied to the two ends of a massless, inextensible string passing over a smooth pulley. When the system is released, calculte the acceleration of the masses and the tension in the string.

Solution 23

Let the acceleration of the masses and the tension in the string be a and T respectively. The value of acceleration due to gravity on the surface of the Earth is $g = 9.8$ m/s^2. The schematic of the problem is shown in Figure 5.

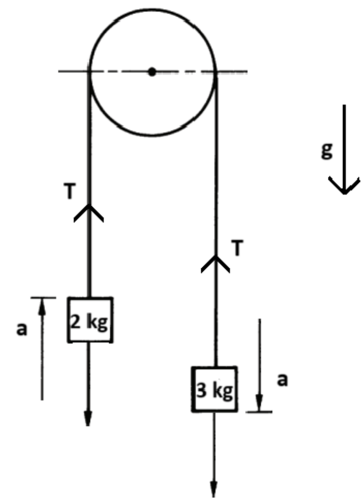

Figure 5: Schematic of Porblem 23

For 2kg mass, the equation of motion is:

$$T - 2g = 2a \qquad \Rightarrow \qquad T - 19.6 = 2a \tag{23.1}$$

For 3kg mass, the equation of motion is:

$$3g - T = 3a \qquad \Rightarrow \qquad 29.4 - T = 3a \tag{23.2}$$

We have two equations of motion and two constants. Hence the system of equations is deterministic. Adding Eqs. (23.1) and (23.2), we get:

$$\boxed{a = 1.96 \text{ m/s}^2} \tag{23.3}$$

Substituting Eq. (23.3) in Eq. (23.1), we get:

$$\boxed{T = 23.52 \text{ N}}$$

Problem 24

A spring of length L_a (no tension) is connected to a support at one end and has a mass M attached at the other. Neglect the mass of the spring, the dimension of the mass M, and assume that the motion is confined to a vertical plane. Also, assume that the spring only stretches without bending but it can swing in the plane.

(a) Using the angular displacement of the mass from the vertical and the length that the string has stretched from its rest length (hanging with the mass M), find Lagrange's equations.

(b) Solve these equations for small stretching and angular displacements.

(c) Solve the equations in part (a) to the next order in both stretching and angular displacement. This part is amenable to hand calculations. Using some reasonable assumptions about the spring constant, the mass, and the rest length, discuss the motion. Is a resonance likely under the assumptions stated in the problem?

(d) (For analytic computer programs.) Consider the spring to have a total mass $m \ll M$. Neglecting the bending of the spring, set up Lagrange's equations correctly to first order in m and the angular and linear displacements.

(e) (For numerical computer analysis.) Make sets of reasonable assumptions of the constants in part (a) and make a single plot of the two coordinates as functions of time.

Solution 24: *This solution is still in progress. I attempted the following approach:*

The spring-mass system is confined to a two dimensional vertical plane. The system has two degrees of freedom and we can describe the system in polar coordinates (r, θ), where r describes position of the mass w.r.t the point of suspension and θ describes the angle the spring makes w.r.t the vertical axis passing through the point of suspension.

(a) The kinetic energy T of the system is:

$$\begin{aligned} T &= \frac{1}{2}Mv^2 \\ &= \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) \end{aligned} \tag{24.1}$$

Let the point of suspension of spring be our reference for potential energy. The potential energy V of the system is:

$$V = -Mgr\cos\theta + \frac{1}{2}k(r - L_a)^2 \qquad (k \equiv \text{spring constant}) \tag{24.2}$$

Using Eqs. (24.1) and (24.2), the Lagrangian L of the system can be written as:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) + Mgr\cos\theta - \frac{1}{2}k(r - L_a)^2 \end{aligned} \tag{24.3}$$

Using Eq. (24.3), Euler-Lagrange equation can be solved for our generalized coordinates, yielding the equations of motion as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \Rightarrow \quad \boxed{\ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{k}{M}(r - L_a) = 0} \quad \text{and} \tag{24.4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad \boxed{2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + gr\sin\theta = 0} \tag{24.5}$$

(b) Let the static equilibrium length of the spring be l. Since $\theta = \dot{\theta} = \dot{r} = \ddot{r} = 0$ at static equilibrium, we have using Eq. (24.4):

$$\frac{k}{M}(l - L_a) = g \quad \Rightarrow \quad l = \frac{gM}{k} + L_a \tag{24.6}$$

Let the small radial displacement be $\delta = r - l$ with $\delta \ll l$. Let the angular displacement θ be small. Using these small displacements δ and θ in Eqs. (24.4) and (24.5), and then neglecting negligible terms like $r\dot{\theta}^2$ and $2r\dot{r}\dot{\theta}$, we get:

$$\ddot{\delta} + \frac{k}{M}\delta = 0 \quad \text{and} \quad \left(\because r = \delta + l = \delta + \frac{gM}{k} + L_a \text{ and } \cos\theta \approx 1\right) \tag{24.7}$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad \text{respectively} \quad \left(\because r^2 = (\delta + l)^2 \approx l^2 \text{ and } \sin\theta \approx \theta\right) \tag{24.8}$$

The solutions for Eqs. (24.7) and (24.8) will be:

$$\delta(t) = A\cos(\omega_r t + \phi_r) \quad ; \quad \omega_r = \sqrt{\frac{k}{M}} \quad , \quad \phi_r = \text{constant} \quad \text{and} \tag{24.9}$$

$$\boxed{\theta(t) = A\cos(\omega_\theta t + \phi_\theta) \quad ; \quad \omega_\theta = \sqrt{\frac{g}{l}} \quad , \quad \phi_\theta = \text{constant}} \quad \text{respectively} \tag{24.10}$$

Using $r(t) = \delta(t) + l$, Eq. (24.9) can be written as:

$$\boxed{r(t) = A\cos(\omega_r t + \phi_r) + l \quad ; \quad \omega_r = \sqrt{\frac{k}{M}} \quad , \quad \phi_r = \text{constant} \quad , \quad l = \frac{gM}{k} + L_a} \tag{24.11}$$

(c) Using small radial displacement $\delta = r - l$ with $\delta \ll l$ and small angular displacement and expanding (24.4) up to quadratic power and neglicting higher power negligible terms, we get:

$$\ddot{\delta} - l\dot{\theta}^2 - g(\cos\theta - 1) + \frac{k}{M}\delta = 0 \qquad (\because r = \delta + l \approx l)$$

$$\Rightarrow \quad \ddot{\delta} - l\dot{\theta}^2 + \frac{g}{2}\theta^2 + \frac{k}{M}\delta = 0 \qquad \text{(neglecting higher order terms in cosine series)}$$

$$\Rightarrow \quad \ddot{\delta} + \omega_r\delta = l\dot{\theta}^2 - \frac{g}{2}\theta^2 \quad ; \quad \omega_r = \sqrt{\frac{k}{M}}$$